

Hausdorff measure and Hausdorff dimension

Brody Dylan Johnson

St. Louis University

Outline:

- Measure theory (3 slides)
- Hausdorff measure (5 slides)
 - @ Construction
 - @ Properties
- Hausdorff dimension (1 slide)
- Examples (7 slides)
 - @ Line segment in \mathbb{R}^2
 - @ Cantor ternary set
- Hausdorff measurable sets (6 slides)
- More examples w/o proof (1 slide)

Measure theory: (1 of 3)

Ingredients:

- X – a space
- \mathcal{A} – a collection of subsets X ;
- $\mu : \mathcal{A} \rightarrow [0, \infty]$ which quantifies the “measure”

Definition 1. We say that (X, \mathcal{A}, μ) is a *measure space* if \mathcal{A} is a σ -algebra (closed under complementation and countable unions) and μ satisfies

i. $\mu(\emptyset) = 0$;

ii. if $\{E_j\}_{j \in \mathbb{N}}$ is a collection of disjoint sets in \mathcal{A} , then

$$\mu \left(\bigcup_{j \in \mathbb{N}} E_j \right) = \sum_{j \in \mathbb{N}} \mu(E_j). \quad (\text{disjoint additivity})$$

Measure theory: (2 of 3)

Definition 2. An *outer measure* on X is a nonnegative set function μ^* (defined for any subset of X) satisfying the following properties:

- i. $\mu^*(\emptyset) = 0$;
- ii. if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$; (monotonicity)
- iii. if $\{E_j\}_{j \in \mathbb{N}}$ is a collection of sets in $\mathcal{P}(X)$ then

$$\mu^* \left(\bigcup_{j \in \mathbb{N}} E_j \right) \leq \sum_{j \in \mathbb{N}} \mu^*(E_j). \quad (\text{subadditivity})$$

Definition 3. A subset $E \subseteq X$ is μ^* -*measurable* if for any $A \subseteq X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Measure theory: (3 of 3)

Theorem 1. *Let μ^* be an outer measure on a space X , \mathcal{A} the collection of μ^* -measurable sets, and $\mu = \mu^*|_{\mathcal{A}}$. Then*

i. \mathcal{A} is a σ -algebra;

ii. (X, \mathcal{A}, μ) is a complete measure space.

Example 1. Let $X = \mathbb{R}$ and define μ^* by

$$\mu^*(E) = \inf \left\{ \sum_{j \in \mathbb{N}} \ell(I_j) : E \subseteq \bigcup_{j \in \mathbb{N}} I_j, I_j \text{ an interval} \right\},$$

where $\ell(I)$ is the usual length of the interval I . Then μ^* is an outer measure and, by way of the previous theorem, gives rise to μ , the Lebesgue measure on \mathbb{R} . This measure is defined on a σ -algebra containing the open sets and retains the property: $\mu(I) = \ell(I)$ for intervals I .

Hausdorff measure: (1 of 5)

Let (X, ρ) be a metric space. Fix $\alpha > 0$ and let $\varepsilon > 0$. Define $\lambda_\alpha^{(\varepsilon)} : \mathcal{P}(X) \rightarrow [0, \infty]$ by

$$\lambda_\alpha^{(\varepsilon)}(E) = \inf \left\{ \sum_{j \in \mathbb{N}} r_j^\alpha : \{B_{x_j}(r_j)\}_{j \in \mathbb{N}} \text{ covers } E, r_j < \varepsilon \right\}.$$

Observe that covers with smaller radii are included in covers that allow larger radii. This implies that if $\varepsilon_1 < \varepsilon_2$ then

$$\lambda_\alpha^{(\varepsilon_1)}(E) \geq \lambda_\alpha^{(\varepsilon_2)}(E).$$

Definition 4. Let $E \subseteq X$. Define $m_\alpha^*(E)$ by

$$m_\alpha^*(E) = \sup \{ \varepsilon > 0 : \lambda_\alpha^{(\varepsilon)}(E) \} = \lim_{\varepsilon \rightarrow 0^+} \lambda_\alpha^{(\varepsilon)}(E).$$

Hausdorff measure: (2 of 5)

Proposition 2. *For $\alpha > 0$, m_α^* is an outer measure.*

Proof. For each $\varepsilon > 0$, $\lambda_\alpha^{(\varepsilon)}(\emptyset) = 0$ and thus $m_\alpha^*(\emptyset) = 0$. Similarly, the monotonicity of $\lambda_\alpha^{(\varepsilon)}$ is inherited by m_α^* . It is not difficult to show that $\lambda_\alpha^{(\varepsilon)}$ is subadditive. Let E be the union of $\{E_j\}_{j \in \mathbb{N}}$ and suppose $m_\alpha^*(E) < \infty$. Thus, there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then

$$m_\alpha^*(E) - \delta < \lambda_\alpha^{(\varepsilon)}(E) \leq m_\alpha^*(E).$$

This implies

$$m_\alpha^*(E) < \lambda_\alpha^{(\varepsilon)}(E) + \delta \leq \sum_{j \in \mathbb{N}} \lambda_\alpha^{(\varepsilon)}(E_j) + \delta \leq \sum_{j \in \mathbb{N}} m_\alpha^*(E_j) + \delta.$$

Since $\delta > 0$ was arbitrary this yields the desired inequality. (The case in which $m_\alpha^*(E) = \infty$ is similar.) \square

Hausdorff measure: (3 of 5)

Definition 5. For $\alpha > 0$, the restriction of m_α^* to the m_α^* -measurable sets is called the *Hausdorff measure of dimension α* for (X, ρ) and is denoted by m_α .

A natural question is:

What are the m_α^* -measurable subsets of (X, ρ) ?

Proposition 3 (Proposition (12.41) [1]). *Let (X, ρ) be a metric space and let μ^* be an outer measure on X with the property that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$. Then every closed set (hence every Borel set) is measurable with respect to μ^* .*

(Proof postponed.)

Hausdorff measure: (4 of 5)

Proposition 4. *Let $\alpha > 0$ and (X, ρ) a metric space. If $A, B \subseteq X$ satisfy $\rho(A, B) > 0$ then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.*

Proof. The inequality \leq follows from the subadditivity property. For the reverse inequality, let $\rho(A, B) = 2\delta > 0$ and observe that if $\varepsilon < \delta$ then

$$\lambda_{\alpha}^{(\varepsilon)}(A \cup B) \geq \lambda_{\alpha}^{(\varepsilon)}(A) + \lambda_{\alpha}^{(\varepsilon)}(B).$$

This stems from the fact that a ball of radius less than ε which intersects A cannot meet a ball of radius less than ε which intersects B . Letting $\varepsilon \rightarrow 0+$, this yields

$$m_{\alpha}^*(A \cup B) \geq m_{\alpha}^*(A) + m_{\alpha}^*(B).$$

□

CONCLUSION: All Borel sets in (X, ρ) are m_{α}^* -measurable.

Hausdorff measure: (5 of 5)

Proposition 5 (Exercise 12.55, [1]). *Let $\alpha > 0$ and (X, ρ) a metric space. Then for any Borel set E ,*

i. If $m_\alpha(E) < \infty$ then $m_\beta(E) = 0$ for all $\beta > \alpha$;

ii. If $m_\alpha(E) > 0$ then $m_\beta(E) = \infty$ for all $0 < \beta < \alpha$.

Proof. Let $\{B_j(r_j)\}_{j \in \mathbb{N}}$ be a cover of E with $r_j < \varepsilon$, $j \in \mathbb{N}$. Suppose $m_\alpha(E) < \infty$ and $\beta > \alpha$, then

$$\sum_{j \in \mathbb{N}} r_j^\beta = \sum_{j \in \mathbb{N}} r_j^\alpha r_j^{\beta - \alpha} \leq \varepsilon^{\alpha - \beta} \sum_{j \in \mathbb{N}} r_j^\alpha.$$

This implies $\lambda_\alpha^{(\varepsilon)}(E) \leq \varepsilon^{\beta - \alpha} \lambda_\alpha^{(\varepsilon)}(E)$ and, moreover, that

$$m_\beta(E) = \lim_{\varepsilon \rightarrow 0^+} \lambda_\beta^{(\varepsilon)}(E) \leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\beta - \alpha} \lambda_\alpha^{(\varepsilon)}(E) = 0.$$

This completes the proof of (i). The proof of (ii) is analogous. \square

Hausdorff dimension: (1 of 1)

Definition 6. Let E be a Borel subset of (X, ρ) . The *Hausdorff dimension* of E is defined as $\dim_H(E)$, given by

$$\begin{aligned}\dim_H(E) &= \inf \{ \alpha > 0 : m_\alpha(E) = 0 \} \\ &= \sup \{ \alpha \geq 0 : m_\alpha(E) = \infty \}.\end{aligned}$$

In light of the previous proposition the notion of Hausdorff dimension is well-defined.

MINI PROBLEM: Show that the Hausdorff dimension of a singleton is zero.

Example 1: (1 of 2)

Let $X = \mathbb{R}^2$ with the Euclidean metric and let E be given by

$$E = \{(t, t) : 0 \leq t \leq 1\}.$$

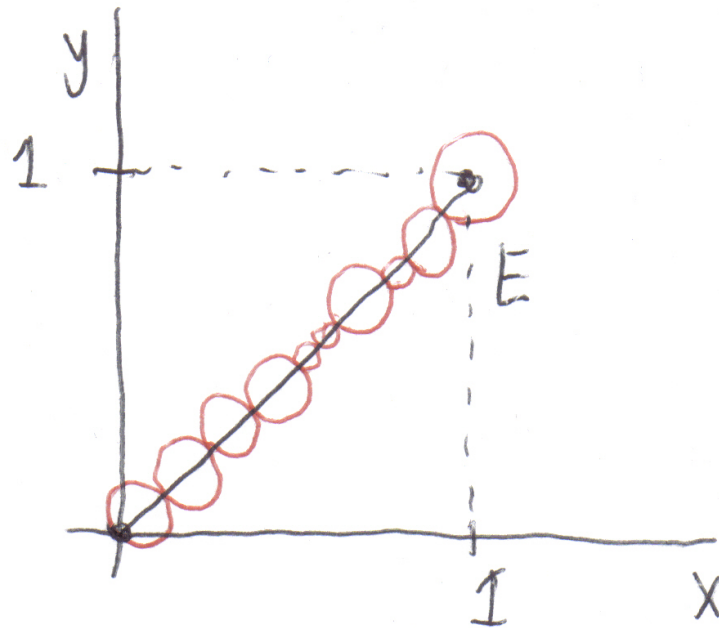


Figure 1: A representative cover of E .

Example 1: (2 of 2)

It is reasonable to expect $\dim_H(E) = 1$ and if it turns out that $0 < m_1^*(E) < \infty$ then this expectation will be confirmed. Let $\varepsilon > 0$ and observe that

$$\lambda_1^{(\varepsilon)}(E) = \inf \left\{ \sum_{j \in \mathbb{N}} r_j : \{B_j\}_{j \in \mathbb{N}} \text{ covers } E, r_j < \varepsilon \right\}.$$

Observe that the sum of the radii is half the sum of the diameters, which can be made arbitrarily close to the length of the segment. (as depicted above)

CONCLUSION: For $\varepsilon > 0$, $\lambda_1^{(\varepsilon)}(E)$ is half the length of the curve and thus $0 < m_1^*(E) < \infty$, i.e., $\dim_H(E) = 1$.

Example 2: (1 of 5)

Let $X = \mathbb{R}$ with the Euclidean metric and let E be the Cantor ternary set. Let

$$\begin{aligned} E_1 &= [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\ E_2 &= E_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right) \\ &\vdots \end{aligned}$$

where E_n is constructed from E_{n-1} by deleting the middle-third of each closed interval comprising E_{n-1} . The Cantor set is thus

$$E = \bigcap_{n \in \mathbb{N}} E_n$$

Example 2: (2 of 5)

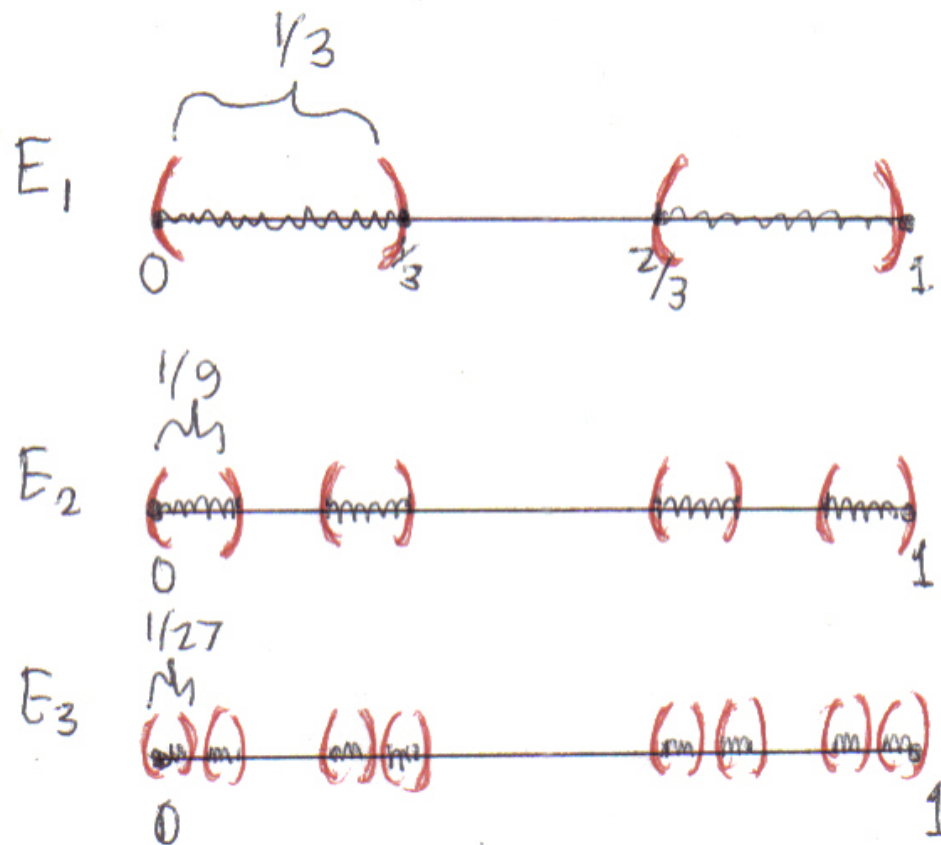


Figure 2: Depictions of E_n and corresponding covers.

Example 2: (3 of 5)

Proposition 6. *The Cantor ternary set E satisfies*

$$\dim_H(E) \leq \log_3 2.$$

Sketch of proof: Notice that E_n may be covered by 2^n closed intervals of length 3^{-n} and recall that $E \subseteq E_n$. Letting $\varepsilon_n = \frac{1}{2}3^{-n}$, then

$$\lambda_\alpha^{(\varepsilon_n)}(E) \leq \sum_{j=1}^{2^n} \left(\frac{3^{-n}}{2} \right)^\alpha = 2^{-\alpha} (2^n 3^{-n\alpha}).$$

If $\alpha > \log_3 2$ then $3^\alpha > 2$ so $\lambda_\alpha^{(\varepsilon_n)}(E) \rightarrow 0$ as $n \rightarrow \infty$, implying $m_\alpha(E) = 0$ if $\alpha > \log_3 2$. □

Example 2: (4 of 5)

Proposition 7. *The Cantor ternary set E satisfies*

$$\dim_H(E) = \log_3 2.$$

Sketch of sketch of proof: This argument comes from:

http://en.wikipedia.org/wiki/Hausdorff_dimension .

Let $\psi_1(x) = \frac{1}{3}x$ and $\psi_2(x) = \frac{1}{3}x + \frac{2}{3}$. These maps are contractive maps with the property that

$$E = \psi_1(E) \cup \psi_2(E). \quad (\text{i.e., } E \text{ is self-similar})$$

Thus, E is a fixed point of the the contractive mapping

$$A \mapsto \psi_1(A) \cup \psi_2(A)$$

and under certain technical conditions (which apply here) E is the unique such fixed point.

Example 2: (5 of 5)

Under another technical assumption on the contractive maps and given that each contraction is the composition of an isometry and a dilation about some point, it has been proven that the unique fixed point of $A \mapsto \psi_1(A) \cup \psi_2(A)$ has Hausdorff dimension s satisfying

$$\left(\frac{1}{r_1}\right)^s + \left(\frac{1}{r_2}\right)^s = 1,$$

where r_1, r_2 are the dilations of ψ_1, ψ_2 .

Thus, the dimension of the Cantor set must satisfy

$$2 \left(\frac{1}{3}\right)^s = 1 \quad \text{or} \quad s = \frac{\log 2^{-1}}{\log 3^{-1}} = \log_3 2. \quad \square$$

Hausdorff measurable sets: (1 of 6)

Let (X, ρ) be a metric space and Λ a collection of \mathbb{R} -valued functions on X .

Definition 7. An outer measure μ^* is called a *Carathéodory outer measure* with respect to Λ if:

Given $A, B \subseteq X$ which are separated by $\varphi \in \Lambda$, it follows that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Here, $A, B \subseteq X$ are *separated* by $\varphi \in \Lambda$ if there exist $a, b \in \mathbb{R}$, $a > b$ such that $\varphi > a$ on one of A, B and $\varphi < b$ on the other set.

Hausdorff measurable sets: (2 of 6)

Proposition 8 (Proposition (12.40) [1]). *If μ^* is a Carathéodory outer measure with respect to Λ , then every function in Λ is μ^* -measurable.*

Proof: It will be shown that $E = \{x : \varphi(x) > a\}$ is μ^* -measurable, which would follow from

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c), \quad A \subseteq X.$$

Without loss of generality, assume $\mu^*(A) < \infty$ and set $B = A \cap E$, $C = A \cap E^c$. Let $B_n = \{x : x \in B \text{ and } \varphi(x) > a + \frac{1}{n}\}$ and notice that $\{B_n\}_{n \in \mathbb{N}}$ is an increasing sequence. Define $R_n = B_n \setminus B_{n-1}$ so that

$$B = B_n \cup \left(\bigcup_{k>n} R_k \right).$$

Hausdorff measurable sets: (3 of 6)

1. φ separates B_{n-2} and R_n . If $x \in B_{n-2}$ then $\varphi(x) > a + \frac{1}{n-2}$. If $x \in R_n$ then $x \notin B_{n-1}$ so $\varphi(x) \leq a + \frac{1}{n-1}$.

2. φ separates $\cup_{j=1}^{k-1} R_{2j}$ and R_{2k} . This follows from the fact that $\cup_{j=1}^{k-1} R_{2j} \subseteq B_{2k-2}$ which is separated from R_{2k} by Step 1.

3. Step 2 together with the fact that μ^* is Carathéodory allows the inductive calculation,

$$\mu^*\left(\cup_{j=1}^k R_{2j}\right) = \mu^*(R_{2k}) + \mu^*\left(\cup_{j=1}^{k-1} R_{2j}\right) = \sum_{j=1}^k \mu^*(R_{2j}).$$

4. Recall that $\cup_{j=1}^k R_{2j} \subseteq B \subseteq A$, so $\sum_{j=1}^k \mu^*(R_{2j}) \leq \mu^*(A)$. It follows that that $\sum_{j \in \mathbb{N}} \mu^*(R_{2j})$ is convergent.

Hausdorff measurable sets: (4 of 6)

4. (continued) Analogous reasoning shows that $\sum_{j \in \mathbb{N}} \mu^*(R_{2j+1})$ is convergent and, therefore, $\sum_{j \in \mathbb{N}} \mu^*(R_j)$ is finite.

5. Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\sum_{k \geq n} \mu^*(R_k) < \varepsilon$. Subadditivity of μ^* yields

$$\mu^*(B) \leq \mu^*(B_n) + \sum_{k \geq n+1} \mu^*(R_k) < \mu^*(B_n) + \varepsilon.$$

6. Recall that $\varphi > a + \frac{1}{n}$ on B_n and $\varphi \leq a$ on $C = A \cap E^c$, so φ separates B_n and C . Thus,

$$\mu^*(A) \geq \mu^*(B_n \cup C) = \mu^*(B_n) + \mu^*(C) > \mu^*(B) + \mu^*(C) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary the desired inequality follows. \square

Hausdorff measurable sets: (5 of 6)

It is now possible to prove Proposition 3, which states that if μ^* satisfies

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

whenever $\rho(A, B) > 0$, then every closed set is μ^* -measurable. Our goal is to prove that μ^* is a Carathéodory outer measure in order to apply Proposition 8.

Proof of Proposition 3:

1. Let $\Lambda = \{f : X \rightarrow \mathbb{R} : f(x) = \rho(x, E), E \subseteq X\}$ and suppose $\varphi \in \Lambda$ separates $A, B \subseteq X$. Without loss of generality, assume $a > b$, $\varphi|_A > a$, and $\varphi|_B < b$. Because $\varphi \in \Lambda$, there is $E \subseteq X$ such that $\varphi(x) = \rho(x, E)$.

Hausdorff measurable sets: (6 of 6)

2. Let $\varepsilon = \frac{1}{2}(a - b)$. Suppose by way of contradiction that $\rho(A, B) = 0$, then there exist $x \in A$ and $y \in B$ such that $\rho(x, y) < \varepsilon$. Notice, however that

$$\rho(x, E) \leq \rho(x, y) + \rho(y, E) < \frac{1}{2}(a - b) + b = \frac{1}{2}(a + b) < a,$$

since $\rho(y, E) = \varphi(y) < b$. But this means $a > \rho(x, E) = \varphi(x) > a$, which is impossible.

3. Therefore, by the hypothesis, A, B separated by Λ implies $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. Proposition 8 then implies that every function in Λ is μ^* -measurable. If $F \subseteq X$ is closed then

$$F = \{x : \rho(x, F) \leq 0\},$$

so F is μ^* measurable since $\rho(x, F) \in \Lambda$. \square

More examples: (1 of 1)

Source:

http://en.wikipedia.org/wiki/Hausdorff_dimension

- The Sierpinski gasket has Hausdorff dimension $\log_2 3$.
- Brownian motion in dimension two or greater has Hausdorff dimension 2.
- The Peano curve has Hausdorff dimension 2.

References

- [1] H. L. Royden. *Real Analysis*. Prentice-Hall, 1988.