

## TRIGONOMETRIC INTERPOLATION

ABSTRACT. Following Chapter 10 of [Zygmund], these notes investigate the subject of trigonometric interpolation. It will be shown that the  $k$ th partial sum of the  $n$ th interpolating polynomial,  $I_{n,k}[f]$ , minimizes the square error with respect to a certain Riemann-Stieltjes integral among trigonometric polynomials of degree  $k$ .

### 1. INTRODUCTION

A trigonometric polynomial of order  $n$  takes the form

$$T(x) = \sum_{k=-n}^n c_k e^{2\pi i k x}, \tag{1}$$

and has  $2n + 1$  coefficients,  $c_k \in \mathbb{C}$ .

**Theorem 1** (Theorem X.1.2 of [Zygmund]). *Given  $2n + 1$  distinct points  $x_0, x_1, \dots, x_{2n}$  and arbitrary numbers  $y_0, y_1, \dots, y_{2n}$  (real or complex), there is always a unique trigonometric polynomial  $T(x)$  such that*

$$T(x_k) = y_k, \quad 0 \leq k \leq 2n.$$

*Proof.* The result can be expressed as a matrix equation,

$$\begin{bmatrix} e^{-2\pi i n x_0} & \dots & e^{2\pi i n x_0} \\ \vdots & \ddots & \vdots \\ e^{-2\pi i n x_{2n}} & \dots & e^{2\pi i n x_{2n}} \end{bmatrix} \begin{pmatrix} c_{-n} \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_{2n} \end{pmatrix},$$

so it is sufficient to prove that the determinant of the  $(2n + 1) \times (2n + 1)$  matrix on the left-hand side is nonzero. Notice that this determinant is equivalent to

$$e^{-2\pi i n(x_0+x_1+\dots+x_{2n})} \begin{vmatrix} 1 & e^{2\pi i x_0} & \dots & e^{2\pi i 2n x_0} \\ 1 & e^{2\pi i x_1} & \dots & e^{2\pi i 2n x_1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2\pi i x_{2n}} & \dots & e^{2\pi i 2n x_{2n}} \end{vmatrix} = e^{-2\pi i n(x_0+x_1+\dots+x_{2n})} \prod_{0 \leq j < k \leq 2n} (e^{2\pi i x_k} - e^{2\pi i x_j}),$$

by virtue of the fact that the matrix above is a Vandermonde matrix. Given that the  $x_k$ 's are distinct, this determinant is nonzero. □

Theorem 1 implies the existence of  $2n + 1$  trigonometric polynomials,  $t_j(x)$ ,  $0 \leq j \leq 2n$ , such that  $t_j(x_k) = \delta_{j,k}$ . One can then solve the above interpolation problem by writing

$$T(x) = \sum_{j=0}^{2n} y_j t_j(x).$$

In particular, observe that these *fundamental polynomials* take the form

$$t_j(x) = \frac{\prod_{k \neq j} 2 \sin(\pi(x - x_k))}{\prod_{k \neq j} 2 \sin(\pi(x_j - x_k))} = \prod_{k \neq j} (\alpha_k e^{\pi i x} + \beta_k e^{-\pi i x}),$$

since the above definition satisfies  $t_j(x+1) = t_j(x)$  and is a product of  $2n$  terms of the form  $\alpha_k e^{\pi i x} + \beta_k e^{-\pi i x}$ , guaranteeing that the degree of  $t_j$  is no more than  $n$ .

**Theorem 2** (Theorem X.1.7 of [Zygmund]). *The number of roots of any  $T(x) \not\equiv 0$  of order  $n$  does not exceed  $2n$ .*

*Proof.* Notice that  $P(z) = e^{-2\pi i n x} T(x)$  is an ordinary polynomial of degree  $2n$  in  $z = e^{2\pi i x}$ . Moreover,  $x_0$  is root of  $T(x)$  of order  $k$  if and only if  $z_0 = e^{2\pi i x_0}$  is a root of order  $k$  of  $P(z)$ . Thus, if the number of roots of  $T(x)$  exceeds  $2n$  so would the number of roots of  $P(z)$ , which would imply  $T(x) \equiv 0$  by the fundamental theorem of algebra.  $\square$

This implies the following statement:

*If  $S(x)$  and  $T(x)$  are two trigonometric polynomials of order  $n$  which vanish at the same  $2n$  points  $x_1, x_2, \dots, x_{2n} \in \mathbb{T}$  (where roots of multiplicity occur when some of the points coincide), then there is a scalar  $\alpha$  such that  $S(x) = \alpha T(x)$ .*

Indeed, if  $S \not\equiv 0$  then there is a scalar  $\alpha$  and  $\xi \in \mathbb{T}$  such that  $S(\xi) \neq 0$ , but  $S(\xi) - \alpha T(\xi) = 0$ . Then  $S(x) - \alpha T(x)$  vanishes at  $2n + 1$  points and Theorem 2 implies  $S(x) = \alpha T(x)$ .

## 2. THE RIEMANN-STIELTJES INTEGRAL

This section follows §6-3 of [Kirkwood]. Let  $f$  be a bounded function on  $[a, b]$ , let  $g$  be a nondecreasing function on  $[a, b]$ , and let  $\mathbb{P}$  be the collection of partitions of  $[a, b]$ . If

$$S_+(f) := \inf \{S_+(f; P) : P \in \mathbb{P}\} = \sup \{S_-(f; P) : P \in \mathbb{P}\} =: S_-(f),$$

then  $f$  is *integrable with respect to  $g$*  on  $[a, b]$ . In this case the *Riemann-Stieltjes integral of  $f$  with respect to  $g$*  is denoted by

$$\int_a^b f(x) dg = S_+(f) = S_-(f).$$

Here,  $S_-$  and  $S_+$ , respectively, represent the usual notions of lower and upper sums, i.e.,

$$S_+(f; P) = \sum_{k=1}^n M_k(f)[g(x_k) - g(x_{k-1})] \quad \text{and} \quad S_-(f; P) = \sum_{k=1}^n m_k(f)[g(x_k) - g(x_{k-1})].$$

Many of the basic properties of Riemann sums carry over to Riemann-Stieltjes sums.

**Theorem 3** (Theorem 6-21 of [Kirkwood]). *Let  $f$  and  $g$  be as above.*

(a) *If  $Q$  is a refinement of the partition  $P$ , then*

$$S_+(f; Q) \leq S_+(f; P) \quad \text{and} \quad S_-(f; Q) \geq S_-(f; P).$$

(b) *For any partitions  $P$  and  $Q$  of  $[a, b]$ ,*

$$S_-(f; P) \leq S_+(f; Q).$$

Moreover, the Riemann condition for integrability also holds, i.e.,  $f$  is integrable with respect to  $g$  if and only if, given  $\varepsilon > 0$ , there is a partition  $P(\varepsilon)$  such that

$$S_+(f; P(\varepsilon)) - S_-(f; P(\varepsilon)) < \varepsilon.$$

**Theorem 4** (Theorem 6-24 of [Kirkwood]). *If  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable with respect to  $g$  on  $[a, b]$ .*

As an example, suppose that  $g$  is defined to be zero on  $[0, \frac{1}{2}]$  and one on  $[\frac{1}{2}, 1]$ . Given any continuous function  $f$  on  $[0, 1]$  observe that

$$S_+(f; P) = M_j(f) \quad \text{and} \quad S_-(f; P) = m_j(f),$$

where  $\frac{1}{2} \in [x_{j-1}, x_j]$ . Since  $f$  is assumed continuous each of these quantities can be made arbitrarily close to  $f(\frac{1}{2})$  and thus

$$\int_0^1 f(x) dg = f(1/2).$$

It will be useful to construct a particular Riemann-Stieltjes integral for use in the next section. For  $n \in \mathbb{N}$ , let  $g(x)$ ,  $x \in [0, 1]$ , be a step function equal to  $\frac{j}{2n+1}$  on  $[j/(2n+1), (j+1)/(2n+1))$  for  $0 \leq j \leq 2n$  with  $g(1) = 1$ . Now fix  $x_0 \in [0, 1)$  and consider the points  $x_j \in \mathbb{T} \simeq [0, 1)$  given by

$$x_j = x_0 + \frac{j}{2n+1}, \quad 0 \leq j \leq 2n+1. \tag{2}$$

If  $f$  is a continuous function on  $\mathbb{T}$ , the Riemann-Stieltjes integral of  $f(\cdot - x_0)$  with respect to  $g$  will be denoted by

$$\int_{\mathbb{T}} f(x) d\omega_{x_0}(x) := \int_0^1 f(x - x_0) dg(x). \tag{3}$$

It is not difficult to see that

$$\int_{\mathbb{T}} f(x) d\omega_{x_0}(x) = \frac{1}{2n+1} \sum_{j=0}^{2n} f(x_j).$$

### 3. INTERPOLATING POLYNOMIALS

Throughout this section,  $x_j$ ,  $0 \leq j \leq 2n$ , will represent the equidistant nodal points defined by (2). The polynomial coinciding with a given periodic function  $f(x)$  at the points  $x_j$ ,  $0 \leq j \leq 2n$ , will be denoted by  $I_n(x, f)$  or  $I_n[f]$  and will be called the *n*th interpolating polynomial of  $f$ .

Recall that the Dirichlet kernel is the order  $n$  trigonometric polynomial,

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^n \cos 2\pi kx = \frac{\sin 2\pi(n + \frac{1}{2})x}{2 \sin \pi x},$$

which vanishes at the points  $j/(2n+1)$ ,  $1 \leq j \leq 2n$ , and takes the value  $n + \frac{1}{2}$  at  $x = 0$ . It follows that the order  $n$  trigonometric polynomial  $D_n(x - x_j)/(n + \frac{1}{2})$  is equal to 1 when  $x = x_j$  and 0 when for  $x = x_k$ ,  $k \neq j$ . Hence, given equidistant nodal points the fundamental polynomials have the special form  $t_j(x) = D_n(x - x_j)/(n + \frac{1}{2})$  and thus

$$I_n(x, f) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j) D_n(x - x_j).$$

This formula can be expressed using the Riemann-Stieltjes integral constructed in the last section,

$$I_n(x, f) = 2 \int_{\mathbb{T}} f(t) D_n(x - t) d\omega_{x_0}(t),$$

provided that  $f$  is continuous. Given a trigonometric polynomial  $S(x)$  of order  $n$  it follows that  $I_n(x, S) = S(x)$ . Because  $I_n(x, f)$  is an order  $n$  trigonometric polynomial there exist  $c_k$ ,  $|k| \leq n$ , such that

$$I_n(x, f) = \sum_{k=-n}^n c_k e^{2\pi i k x}.$$

By expanding the integral expression involving the Dirichlet kernel one can show that

$$c_k = \int_{\mathbb{T}} f(t) e^{-2\pi i k t} d\omega_{x_0}(t) = \langle f, e^{2\pi i k \cdot} \rangle_{d\omega_{x_0}}.$$

In fact, the collection  $\{e^{2\pi i k \cdot}\}_{|k| \leq n}$  is orthogonal with respect to  $d\omega_{2n+1}$ , i.e.,

$$\langle e^{2\pi i k \cdot}, e^{2\pi i \ell \cdot} \rangle_{d\omega_{x_0}} = \frac{1}{2n+1} \sum_{j=0}^{2n} e^{2\pi i(k-\ell)x_j} = \frac{e^{2\pi i(k-\ell)x_0}}{2n+1} \sum_{j=0}^{2n} \left( e^{2\pi i(k-\ell)/(2n+1)} \right)^j = \delta_{k,\ell},$$

since  $1 + z + \dots + z^{2n} = 0$  for any  $(2n+1)$ th root of unity  $z \neq 1$ .

It is also interesting to consider the  $k$ th partial sum of the interpolating polynomial  $I_n[f]$ , which is defined as

$$I_{n,k}(x, f) = \sum_{|\ell| \leq k} c_\ell e^{2\pi i \ell x} = 2 \int_{\mathbb{T}} f(t) D_k(t-x) d\omega_{x_0}(t), \quad 0 \leq k \leq n.$$

It turns out that  $I_{n,k}[f]$  solves the least-square regression problem for approximation of a periodic function at equidistant nodal points in terms of an order  $n$  trigonometric polynomial.

**Theorem 5** (Theorem X.2.10 of [Zygmund]). *The polynomial  $I_{n,k}[f]$  minimizes the integral*

$$\int_{\mathbb{T}} |f(x) - S(x)|^2 d\omega_{x_0}(x)$$

among trigonometric polynomials  $S$  of order  $k$ .

*Proof.* Let  $T(x) = I_{n,k}[f]$  and consider  $S(x)$  given by

$$S(x) = \sum_{|\ell| \leq k} a_\ell e^{2\pi i \ell x}.$$

Hence,

$$\begin{aligned} \int_{\mathbb{T}} |f(x) - S(x)|^2 d\omega_{x_0}(x) &= \int_{\mathbb{T}} |f(x)|^2 - f(x) \overline{S(x)} - \overline{f(x)} S(x) + |S(x)|^2 d\omega_{x_0}(x) \\ &= \int_{\mathbb{T}} |f(x)|^2 d\omega_{x_0}(x) + \sum_{|\ell| \leq k} (-\overline{a_\ell} c_\ell - a_\ell \overline{c_\ell} + |a_\ell|^2) \\ &= \int_{\mathbb{T}} |f(x)|^2 d\omega_{x_0}(x) - \sum_{|\ell| \leq k} |c_\ell|^2 + \sum_{|\ell| \leq k} |a_\ell - c_\ell|^2 \\ &= \int_{\mathbb{T}} |f(x) - T(x)|^2 d\omega_{x_0}(x) + \sum_{|\ell| \leq k} |a_\ell - c_\ell|^2. \end{aligned}$$

The last term is nonnegative and, therefore,  $T(x)$  yields the least possible square-error at the points  $x_j$ ,  $0 \leq j \leq 2n$ .  $\square$

#### REFERENCES

- [Kirkwood] J. Kirkwood, *An Introduction to Analysis*, 2nd edition, PWS Publishing Company, Boston, (1995).  
 [Zygmund] A. Zygmund, *Trigonometric series*, 2nd edition, Cambridge Univ. Press, London, (1959).